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A topological sigma model of biKähler geometry

Roberto Zucchini

Dipartimento di Fisica, Università degli Studi di Bologna
V. Irnerio 46, I-40126 Bologna, Italy, and
I.N.F.N., sezione di Bologna, Italy
E-mail: zucchinir@bo.infn.it

Abstract: BiKähler geometry is characterized by a riemannian metric g_{ab} and two covariantly constant generally non commuting complex structures ${K_\pm}^a{}_b$, with respect to which g_{ab} is hermitean. It is a particular case of the bihermitean geometry of Gates, Hull and Roceck, the most general sigma model target space geometry allowing for $(2,2)$ world sheet supersymmetry. We present a sigma model for biKähler geometry that is topological in the following sense: *i*) the action is invariant under a fermionic symmetry δ; *ii*) δ is nilpotent on shell; *iii*) the action is δ-exact on shell up to a topological term; *iv*) the resulting field theory depends only on a subset of the target space geometrical data. The biKähler sigma model is obtainable by gauge fixing the Hitchin model with generalized Kähler target space. It further contains the customary A topological sigma model as a particular case. However, it is not seemingly related to the $(2,2)$ supersymmetric biKähler sigma model by twisting in general.

Keywords: BRST Symmetry, Topological Field Theories, Sigma Models, Differential and Algebraic Geometry.

Contents

1. Introduction

Type II superstring Calabi-Yau compactifications are described by $(2, 2)$ superconformal sigma models with Calabi-Yau target manifolds. These field theories are however rather complicated and, so, they are difficult to study. In 1988, Witten showed that a $(2, 2)$ supersymmetric sigma model on a Calabi-Yau space could be twisted in two different ways, to give the so called A and B topological sigma models [1, 2]. Unlike the original untwisted sigma model, the topological models are soluble: the calculation of observables can be reduced to classical problems of geometry. For this reason, the topological sigma models constitute an ideal field theoretic ground for the in depth study of 2-dimensional supersymmetric field theories.

Witten's analysis was restricted to the case where the sigma model target space geometry was Kähler. In a classic paper, Gates, Hull and Rocek [3] showed that, for a 2-dimensional sigma model, the most general target space geometry allowing for $(2, 2)$ supersymmetry was bihermitean or Kähler with torsion geometry. This is characterized by a riemannian metric g_{ab}, two generally non commuting complex structures $K_\pm{}^a{}_b$ and a closed 3-form H_{abc}, such that g_{ab} is hermitean with respect to both the $K_\pm{}^a{}_b$ and the $K_\pm{}^a{}_b$ are parallel with respect to two different metric connections with torsion proportional to $\pm H_{abc}$ [4−7]. This geometry is much more general than that considered by Witten, which corresponds to the case where $K_+{}^a{}_b = \pm K_-{}^a{}_b$ and $H_{abc} = 0$.

In 2002, Hitchin formulated the notion of generalized complex geometry, which at the same time unifies and extends the customary notions of complex and symplectic geometry and incorporates a natural generalization of Calabi-Yau geometry [8]. Hitchin's ideas were developed by Gualtieri [9], who introduced the notion of generalized Kähler geometry and showed that the bihermitean geometry of Gates, Hull and Roceck was equivalent to the latter.

In refs. [10, 11], Kapustin and Kapustin and Li defined and studied the analogues of A and B models for $(2, 2)$ supersymmetric sigma models with H field and showed that the results were naturally expressed in the language of generalized complex geometry. Simultaneously, other attempts were made to construct sigma models based on generalized complex or Kähler geometry, by invoking world sheet supersymmetry, employing the Batalin-Vilkovisky quantization algorithm, etc. [12 − 21]. All these attempts were somehow unsatisfactory either because they remained confined to the analysis of geometrical aspects of the sigma models or because they yielded field theories, which though interesting in their own, were not directly suitable for quantization and showed no apparent kinship with Witten's A and B models.

In this paper, we present a topological sigma model with target space biKähler geometry. This geometry is characterized by a riemannian metric g_{ab} and two covariantly constant generally non commuting complex structures $K_\pm{}^a{}_b$, with respect to which g_{ab} is hermitean. It is a particular case of the bihermitean geometry of ref. [3] corresponding to $H_{abc} = 0$.

The biKähler sigma model expondeded in the paper has all the basic features of a topological sigma model as summarized below.

a. The action S possesses an odd symmetry δ.

b. δ is nilpotent on shell.

c. S is δ-exact on shell up to a topological term, with some weak restrictions on the target space geometry.

d. The resulting field theory depends only on a certain combination of the target space geometrical data.

The model also has other interesting features.

e. It is obtainable by gauge fixing the Hitchin model [19, 20] with generalized Kähler target space [9] according to the general philosophy of Alexandrov, Kontsevich, Schwartz and Zaboronsky [22].

f. In the particular case $K_+{}^a{}_b = -K_-{}^a{}_b$, it reproduces Witten's A topological sigma model [1, 2]. It also yields topological sigma models for product structure and hyperKähler target space geometries.

Roughly speaking, the field content of the model consists of the fields of the A topological sigma model plus a further 1-form field. This latter becomes non propagating and decouples in the particular case of the A model, but it does not in the general case. Also, the algebra of local topological observables is isomorphic to the Poisson-Lichnerowicz cohomology of a certain target space Poisson structure, which is isomorphic to the target space de Rham cohomology in the particular case of the A model, but it is not in the general case.

For these reasons, the biKähler sigma model introduced in the present paper is not seemingly related to the $(2, 2)$ supersymmetric sigma model by twisting in general, at least in the form defined in [11]. This limits its relevance for string theory. However, its very existence is interesting enough from a field theoretic point of view, not least as an exemplification of the methodology of ref. [22].

As is well known, any topological field theory (of cohomological type) describes the intersection theory of a certain moduli space in terms of local quantum field theory. Though we have identified a set of equations, which, based on general arguments of topological field theory, should define the moduli space underlying the biKähler sigma model, we have no geometrical interpretation and no analytic control on it in general. It is possible that the biKähler sigma model found in this paper, though satisfying a number of basic formal prerequisites for a consistent topological field theory, may not pass a closer inspection at the end. The investigation of this matter is left for future work.

The paper is organized as follows. In section 2, we review basic results of biKähler geometry. In section 3, we introduce the biKähler sigma model and present its field content and its action. In section 4, we analyze the symmetries of the model and show that it possesses an odd symmetry δ, that is nilpotent on shell. In section 5, we prove the topological nature of the model by showing that the action is δ-exact on shell up to a topological term, when the target space geometry satisfies certain weak restrictions. We further identify the set of equations describing the underlying moduli space. In section 6, we study the local cohomology of δ and show its relation to Poisson-Lichnerowicz cohomology. In section 7, we write down the action, the symmetries and the moduli space equations of the biKähler sigma model of standard biKähler target geometries and show that the biKähler model contains Witten's A model as a particular case. In section 8, we show that the biKähler model is obtainable by gauge fixing the Hitchin model with generalized Kähler target space and use this result to show that the associated field theory depends only on a certain combination of the target space geometrical data. Finally, in section 9 we compare our results with the elegant geometrical constructions of refs. [10, 11] and discuss briefly the open problems.

2. BiKähler geometry

Let M be a smooth manifold. An almost biKähler structure on M consists of a riemannian metric g_{ab} and two almost complex structures $K_\pm{}^a{}_b$, such that g_{ab} is hermitean with respect to both $K_\pm{}^a{}_b$:

$$K_\pm{}^a{}_c K_\pm{}^c{}_b = -\delta^a{}_b, \tag{2.1}$$

$$K_{\pm ab} + K_{\pm ba} = 0. \tag{2.2}$$

Here and below, indices are raised and lowered by using the metric g_{ab}. An almost biKähler structure on M is a biKähler structure on M if the tensors $K_\pm{}^a{}_b$ are parallel with respect to the Levi-Civita connection ∇_a of g_{ab}

$$\nabla_a K_\pm{}^b{}_c = 0. \tag{2.3}$$

As is well known, this implies that the almost complex structures $K_\pm{}^a{}_b$ are integrable and, thus, that they are complex structures, and that the metric g_{ab} is Kähler with respect to both the $K_\pm{}^a{}_b$. In the following, we consider only biKähler structures.

The complex structures $K_\pm{}^a{}_b$ can be multiplied, being endomorphisms of the tangent bundle TM of M. In this way, they generate an algebra of endomorphisms $\mathcal{A}$. The conventionally normalized anticommutator of the $K_\pm{}^a{}_b$

$$C^a{}_b = \frac{1}{2}(K_+{}^a{}_c K_-{}^c{}_b + K_-{}^a{}_c K_+{}^c{}_b) \tag{2.4}$$

belongs to the center of $\mathcal{A}$. By this fact, it is easy to see that the most general element of the algebra $\mathcal{A}$ is of the form

$$X^a{}_b = Z^a{}_b + \sum_{\alpha=\pm,0} U_\alpha{}^a{}_c K_\alpha{}^c{}_b \tag{2.5}$$

where $K_0{}^a{}_b$ is the conventionally normalized commutator of the $K_\pm{}^a{}_b$

$$K_0{}^a{}_b = \frac{1}{2}(K_+{}^a{}_c K_-{}^c{}_b - K_-{}^a{}_c K_+{}^c{}_b) \tag{2.6}$$

and $Z^a{}_b$ and $U_\alpha{}^a{}_b$, $\alpha = \pm, 0$, are polynomials in $C^a{}_b$.

By (2.2), the $K_{\pm ab}$ are 2-forms. By (2.3), they are parallel and, thus, also closed[1]

$$\partial_{[a} K_{\pm bc]} = 0\,. \tag{2.7}$$

They are in fact the Kähler forms of g_{ab} corresponding to the complex structures $K_\pm{}^a{}_b$. Besides the $K_{\pm ab}$, there is another relevant 2-form in biKähler geometry, K_{0ab}. K_{0ab} is also parallel and, thus, closed

$$\partial_{[a} K_{0bc]} = 0. \tag{2.8}$$

It is not difficult to show that K_{0ab} is of type $(2,0)+(0,2)$ and holomorphic with respect to both complex structures $K_\pm{}^a{}_b$.

Usually, in Kähler geometry, it is convenient to write the relevant tensor identities in the complex coordinates of the underlying complex structure rather than in general coordinates. In biKähler geometry, one is dealing with two generally non commuting complex structures. One could similarly write the tensor identities in the complex coordinates of either complex structures. In this case, however, the convenience of complex versus general coordinates would be limited. We decided, therefore, to opt for general coordinates throughout the paper. To this end, we define the complex tensors

$$\Lambda_\pm{}^a{}_b = \frac{1}{2}\big(\delta^a{}_b - iK_\pm{}^a{}_b\big). \tag{2.9}$$

[1] Here and below, the brackets $[\cdots]$ denote full antisymmetrization of all enclosed tensor indices except perhaps for those between bars $|\cdots|$.

The $\Lambda_\pm{}^a{}_b$ satisfy the relations

$$\Lambda_\pm{}^a{}_c\Lambda_\pm{}^c{}_b = \Lambda_\pm{}^a{}_b, \tag{2.10a}$$
$$\Lambda_\pm{}^a{}_b + \overline{\Lambda}_\pm{}^a{}_b = \delta^a{}_b, \tag{2.10b}$$
$$\Lambda_\pm{}^a{}_b = \overline{\Lambda}_{\pm b}{}^a. \tag{2.10c}$$

Thus, $\Lambda_\pm{}^a{}_b$ are projector valued endomorphisms of the complexified tangent bundle T_cM. The corresponding projection subbundle of T_cM is the $\pm$ holomorphic tangent bundles $T_\pm^{1,0}M$.

The covariant constancy of the complex structures $K_\pm{}^a{}_b$ entails strong restrictions on the Riemann tensor of the Levi-Civita connection,

$$R_{abce}\Lambda_\pm{}^e{}_d = R_{abed}\overline{\Lambda}_\pm{}^e{}_c, \tag{2.11a}$$
$$R_{aecf}\Lambda_\pm{}^e{}_{[b}\Lambda_\pm{}^f{}_{d]} = 0, \tag{2.11b}$$
$$\nabla_f R_{abcg}\Lambda_\pm{}^f{}_{[d}\Lambda_\pm{}^g{}_{e]} = 0 \tag{2.11c}$$

and many other relations following either by complex conjugation or from the known symmetry properties of the Riemann tensor.

There are many interesting examples of biKähler geometries, which will be considered in this paper. A biKähler structure g_{ab}, $K_\pm{}^a{}_b$ satisfying either conditions

$$K_+{}^a{}_b = -K_-{}^a{}_b, \qquad (K) \tag{2.12a}$$
$$K_+{}^a{}_b = K_-{}^a{}_b \qquad (K') \tag{2.12b}$$

is obviously equivalent to an ordinary Kähler structure g_{ab}, $K^a{}_b$, where

$$K^a{}_b = K_-{}^a{}_b. \tag{2.13}$$

Thus, there are two ways a Kähler structure can be embedded into a biKähler structure. The resulting biKähler structures will be called of type K, K' in the following. Conversely, a biKähler structure g_{ab}, $K_\pm{}^a{}_b$ can be viewed as a pair of Kähler structures with the same underlying metric.

More generally, one can consider biKähler structures g_{ab}, $K_\pm{}^a{}_b$ such that

$$K_0{}^a{}_b = 0 \qquad (P) \tag{2.14}$$

(cf. eq. (2.6)). (2.14) is equivalent to the statement that the endomorphisms $K_\pm{}^a{}_b$ commute. We shall call these biKähler structures of type P. For these

$$L^a{}_b = K_+{}^a{}_c K_-{}^c{}_b \tag{2.15}$$

is a riemannian product structure of the manifold M. The manifold M then factorizes locally as a product $M_{+1}\times M_{-1}$ such that the tangent bundles $TM_{\pm1}$ are the ±1 eigenbundles of the endomorphism $L^a{}_b$. From (2.12), it appears that type K, K' biKähler structures are particular cases of type P biKähler structures. The corresponding product structures $L^a{}_b = \pm\delta^a{}_b$ are trivial.

JHEP01(2006)041

Another important class of biKähler structures g_{ab}, $K_\pm{}^a{}_b$ is defined by the condition

$$C^a{}_b = 0. \qquad (HK) \tag{2.16}$$

(cf. eq. (2.4)), which we shall call of type HK. (2.16) is equivalent to the statement that the endomorphisms $K_\pm{}^a{}_b$ anticommute. For these, the endomorphisms

$$K_1{}^a{}_b = K_+{}^a{}_b, \quad K_2{}^a{}_b = K_-{}^a{}_b, \quad K_3{}^a{}_b = K_0{}^a{}_b \tag{2.17}$$

form a hyperKähler structure g_{ab}, $K_i{}^a{}_b$, $i = 1,\ 2,\ 3$. The manifold M then admits a triplet of Kähler structures with the same underlying metric satisfying the quaternion algebra

$$K_i{}^a{}_c K_j{}^c{}_b = -\delta_{ij}\delta^a{}_b + \epsilon_{ijk} K_k{}^a{}_b. \tag{2.18}$$

3. The biKähler sigma model

The biKähler sigma model is a field theoretic realization of biKähler geometry. It is a 2-dimensional sigma model whose target space is a manifold M equipped with a biKähler structure g_{ab}, $K_\pm{}^a{}_b$ and whose world sheet is a Riemann surface Σ, a surface endowed with a complex structure. The fields of the model are the usual embedding field x^a and three further tensor valued form fields y_a, ψ^a, χ_a. They are characterized by their target space and world sheet global properties and by their ghost degree as summarized by the following table.

field	global type	ghost degree	total degree
x^a	$\mathrm{Fun}(\Sigma, M)$	0	0
y_a	$\Omega^{0,0}(\Sigma, x^*\Pi T^*M)$	1	1
ψ^a	$\Omega^{1,0}(\Sigma, x^*\Pi T_-^{0,1}M)$	-1	0
χ_a	$\Omega^{1,0}(\Sigma, x^*T_-^{*0,1}M)$	0	1

(3.1)

Here, Π is the parity reversion operator of vector bundles, which replaces the typical vector fiber with its counterpart of opposite Grassmannality. The total degree of a field is the sum of its world sheet form and ghost degrees and determines its statistics. The fields x^a, y_a are real. The fields ψ^a, χ_a, conversely, are complex. They are conveniently viewed as elements of $\Omega^{1,0}(\Sigma, x^*\Pi T_c M)$, $\Omega^{1,0}(\Sigma, x^*T^*{}_c M)$ satisfying the constraints

$$\psi^a = \overline{\Lambda}_-{}^a{}_b(x)\psi^b, \tag{3.2a}$$

$$\chi_a = \Lambda_-{}_a{}^b(x)\chi_b. \tag{3.2b}$$

Note that these constraints break the symmetry of the target space biKähler geometry with respect to the exchange of the two complex structures $K_\pm{}^a{}_b$. Further, they couple the complex structure $K_-{}^a{}_b$ of M, and the complex structure of Σ, since the target space global properties of the fields involved depend on the latter in an essential way. We shall analyze these issues in greater detail below in sections 8, 9.

The action S of the biKähler sigma model is given by[2]

$$S = \int_\Sigma \Big\{ -(iP^2\overline{\Lambda}_{-ab} + K_{-ab})(x)\overline{\partial}x^a\partial x^b - ig^{ab}(x)\overline{\chi}_a\chi_b \tag{3.3}$$

[2]Here and below, for any number of mixed rank 2 tensors $A_1{}^a{}_b, \ldots, A_p{}^a{}_b$, we set $(A_1+A_2+\cdots+A_p)^a{}_b = A_1{}^a{}_b + A_2{}^a{}_b + \cdots A_p{}^a{}_b$ and $A_1A_2\cdots A_p{}^a{}_b = A_1{}^a{}_{c_1}A_2{}^{c_1}{}_{c_2}\cdots A_p{}^{c_{p-1}}{}_b$. Note laso that $(1)^a{}_b = \delta^a{}_b$, $(1)_{ab} = g_{ab}$, etc.

$$+ (1+P)J^{ab}(x)\big(\overline{\chi}_a + g_{ac}(x)\overline{\partial}x^c\big)\big(\chi_b + g_{bd}(x)\partial x^d\big)$$
$$+ M^a{}_b(x)(\psi^b\overline{\nabla}y_a + \overline{\psi}^b\nabla y_a) + R^a{}_{ced}P^{eb}(x)\overline{\psi}^c\psi^d y_a y_b\Big\},$$

where the tensors $J^a{}_b$, $P^a{}_b$, $M^a{}_b$ are given by

$$J^a{}_b = \frac{1}{2}(K_+ + K_-)^a{}_b, \qquad P^a{}_b = \frac{1}{2}(K_+ - K_-)^a{}_b, \tag{3.4a}$$

$$M^a{}_b = \frac{1}{2}(1+K_+)(1+K_-)^a{}_b \tag{3.4b}$$

and ∇ is the pull-back by x^a of the Levi-Civita connection

$$\nabla = \partial \pm \Gamma^{\cdot}{}_{\cdot a}(x)\partial x^a. \tag{3.5}$$

In (3.3), wedge multiplication of forms is understood. Due the large number of relations satisfied by the basic tensors of biKähler geometry and because of the constraints (3.2), S can be cast in several other equivalent forms. The one shown above is the most compact, which we were able to find.

The classical field equations associated with the action S are easily derived:

$$iP^2{}_{ab}(x)\overline{\nabla}\partial x^b + \nabla_a R^b{}_{dfe}P^{fc}(x)\overline{\psi}^d\psi^e y_b y_c \tag{3.6a}$$
$$+ \Big[R^d{}_{eca}M^e{}_b(x)\overline{\psi}^b\partial x^c + (1+P)J^b{}_a(x)\overline{\nabla}\chi_b + \text{c.c.}\Big] = 0,$$

$$M^a{}_b(x)\overline{\nabla}\psi^b + R^{[a}{}_{dec}P^{|e|b]}(x)\overline{\psi}^c\psi^d y_b + \text{c.c.} = 0, \tag{3.6b}$$

$$M\overline{\Lambda}_-{}^b{}_a(x)\overline{\nabla}y_b + R^b{}_{fed}\overline{\Lambda}_-{}^f{}_a P^{ec}(x)\overline{\psi}^d y_b y_c = 0, \tag{3.6c}$$

$$\overline{\Lambda}_-(1+P)J^a{}_b(x)\overline{\partial}x^b + \overline{\Lambda}_- P(1+J)^{ab}(x)\overline{\chi}_b = 0. \tag{3.6d}$$

In obtaining (3.6c), (3.6d), one must take into due account the constraints (3.2).

The definition of the action S of the biKähler model given in eq. (3.3) may seem rather arbitrary at this stage. It can be justified only *a posteriori* by its remarkable properties shown in the following. Ultimately, these properties can be traced back to just a basic one: the origin of the biKähler model as a gauge fixed form of a suitable generalized Kähler Hitchin model [19, 20], as we shall see in section 8.

The action S may be modified by the addition of topological terms of the form

$$S_{\text{top}} = \int_\Sigma x^*\omega = \int_\Sigma \omega_{ab}(x)\overline{\partial}x^a\partial x^b, \tag{3.7}$$

where ω_{ab} is a closed 2-form, without changing the field equations and the infinitesimal symmetries of the action. For instance

$$\omega_{ab} = \sum_{\alpha=\pm,0} c_\alpha K_{\alpha ab}, \tag{3.8}$$

where the c_α are real coefficients. The terms

$$\int_\Sigma \big(-K_- + (1+P)J\big)_{ab}(x)\overline{\partial}x^a\partial x^b \tag{3.9}$$

appearing in the expression of the action S, eq. (3.3), are precisely of this form. Thus their inclusion is somewhat conventional.

JHEP01(2006)041

4. The symmetries of the model

The biKähler sigma model action S introduced in section 3 exhibits a bosonic symmetry associated with the following infinitesimal even variations

$$\delta_{\rm gh} x^a = 0, \tag{4.1a}$$
$$\delta_{\rm gh} y_a = -y_a, \tag{4.1b}$$
$$\delta_{\rm gh} \psi^a = \psi^a, \tag{4.1c}$$
$$\delta_{\rm gh} \chi_a = 0, \tag{4.1d}$$

where multiplication by an infinitesimal real even parameter is tacitly understood, so that

$$\delta_{\rm gh} S = 0. \tag{4.2}$$

It is easy to see that this nothing but ghost number symmetry. The associated symmetry current is

$$\mathcal{I} = -iM^a{}_b(x)\psi^b y_a + \text{c.c.} \tag{4.3}$$

as is easily verified.

The action S exhibits also a fermionic symmetry associated with the following infinitesimal odd variations

$$\delta x^a = P^{ab}(x) y_b, \tag{4.4a}$$
$$\delta y_a = -\Gamma^b{}_{ad} P^{dc}(x) y_b y_c, \tag{4.4b}$$
$$\delta \psi^a = -\Gamma^a{}_{bd} P^{dc}(x)\psi^b y_c + \overline{\Lambda}_-(J^2+J+1)^a{}_b(x)\partial x^b + \overline{\Lambda}_- P^{ab}(x)\chi_b, \tag{4.4c}$$
$$\delta \chi_a = -\Gamma^b{}_{ad} P^{dc}(x)\chi_b y_c - \Lambda_- J_a{}^b(x)\nabla y_b + \Lambda_{-a}{}^f R^b{}_{fed} P^{ec}(x)\psi^d y_b y_c, \tag{4.4d}$$

where multiplication by an infinitesimal real odd parameter is tacitly understood, so that

$$\delta S = 0. \tag{4.5}$$

The verification of (4.5) is lengthy but totally straightforward. The associated symmetry current is

$$\mathcal{S} = \big(i(J^2+J+1)J - P^2 P\overline{\Lambda}_-\big)^a{}_b(x)\partial x^b p_a + i(P^2+P)J^{ab}(x)\chi_a p_b + \text{c.c.} \tag{4.6}$$

The symmetry δ is nilpotent on shell, as it appears from the following computation

$$\delta^2 x^a = 0, \tag{4.7a}$$
$$\delta^2 y_a = 0, \tag{4.7b}$$
$$\delta^2 \psi^a = -i\overline{\Lambda}_+ \Lambda_-{}^{ga}(x)\Big[M\Lambda_-{}^b{}_g(x)\nabla y_b + R^b{}_{fed}\Lambda_-{}^f{}_g P^{ec}(x)\psi^d y_b y_c\Big], \tag{4.7c}$$
$$\delta^2 \chi_a = R^c{}_{feh}\overline{\Lambda}_-{}^f{}_a P^{hd}(x)\Big[\Lambda_-(1+P)J^e{}_b(x)\partial x^b + \Lambda_- P(1+J)^{eb}(x)\chi_b\Big] y_c y_d \tag{4.7d}$$

and from (3.6c), (3.6d). The verification of (4.7) is also lengthy but straightforward. We note that only two of the four field equations, namely (3.6c), (3.6d), are involved. In more precise terms, (4.7) states that δ is nilpotent on the quotient of the algebra of all field functionals by the bilateral ideal generated by the field equations (3.6c), (3.6d). If we denote by $\approx$ equality on the quotient algebra, we may write

$$\delta^2 \approx 0. \tag{4.8}$$

However, for the sake of brevity, one says simply that δ is nilpotent on shell. The study the cohomology associated with δ is naturally the next step of our analysis.

5. The topological nature of the model

We have seen above that the biKähler sigma model is a sigma model with an odd symmetry that is nilpotent on shell (cf. eqs. (4.5), (4.8)). This makes it akin to some extent to the existent topological models [1, 2]. These latter however have a further property, that is crucial to ensure their topological nature: the action is δ exact on shell up to topological terms. The natural question arises about whether the biKähler sigma model we illustrated above has the same property.

A gauge fermion Ψ is a local functional of the fields of ghost number -1. We are looking for a gauge fermion Ψ such that

$$S \approx \delta\Psi + S_{\rm top}, \tag{5.1}$$

where $S_{\rm top}$ is some topological functional of x^a of the form (3.7) and $\approx$ denotes equality on shell in the sense explained at the end of section 4. A gauge fermion Ψ with the above property exists for the A topological sigma model, as shown by Witten long ago [1, 2]. It is natural to wonder whether a gauge fermion Ψ exists for the biKähler sigma model. We found this problem unexpectedly difficult. In fact, we have not been able to show that such a Ψ exists for an arbitrary biKähler target geometry. However, as we show below, we succeeded in finding a Ψ such that

$$S \approx \delta\Psi + S_{\rm top} + \Omega, \tag{5.2}$$

where Ω is a "topological anomaly", which is generally non vanishing, but which does vanish for a subclass of biKähler structures, defined by a week condition, which moreover contains all the standard examples illustrated in section 2.

Two scenarios are thus possible. In the first scenario, a gauge fermion Ψ satisfying (5.1) exists, but it is rather complicated, making its computation prohibitively difficult. In the second scenario, a gauge fermion Ψ does not exist in general. In such a case, it would be important to characterize the corresponding biKähler structures.

To begin with, we recall that we are tackling a cohomological problem, so that its solution, if it exists, is certainly not unique, but it is affected by the customary cohomological ambiguities. We may start with an ansatz of the form

$$\Psi = \int_\Sigma \frac{i}{2}\Big[A_{ab}(x)(\overline{\psi}^a\partial x^b - \psi^a\overline{\partial}x^b) + B^a{}_b(x)(\overline{\chi}_a\psi^b - \chi_a\overline{\psi}^b)\Big], \tag{5.3}$$

JHEP01(2006)041

where A_{ab}, $B^a{}_b$ are real tensors satisfying

$$\nabla_c A_{ab} = 0, \qquad \nabla_c B^a{}_b = 0. \tag{5.4}$$

Next, we compute $\delta\Psi$ using (4.4) and simplify the resulting expression using the constraints (3.2) and the field equations (3.6c), (3.6d) only. Finally, by a procedure of trial and error, we adjust the expressions of A_{ab}, $B^a{}_b$, in such a way to enforce (5.1) or (5.2). In this way, we find that a relation of the form (5.2) holds if A_{ab}, $B^a{}_b$ are given by

$$A_{ab} = g_{ab} + \frac{1}{Z^2-16}(-1-C+4P)J(1+C+4J-4P)_{ab}, \tag{5.5a}$$

$$B^a{}_b = (1+P)J^a{}_b - \frac{4}{Z^2-16}(1+C)(2+P)J^a{}_b, \tag{5.5b}$$

where

$$C^a{}_b = (J^2-P^2)^a{}_b \tag{5.6}$$

is nothing but the central tensor (2.4) and $Z^a{}_b$ is some function of $C^a{}_b$ subject to the only condition that the endomorphisms $(Z\pm4)^a{}_b$ are pointwise invertible on M. In such a case, the topological term $S_{\rm top}$ is given by

$$\begin{aligned} S_{\rm top} = \int_\Sigma \Big[\frac{1}{2}P(2+J)_{ab}(x) + \frac{1}{2}\Big(\frac{\pm1}{Z\pm4} - \frac{1}{Z^2-16}(C\pm Z+5)\Big) \\ \times P\big(2(-1+C)+(1+C)J\big)_{ab}(x)\Big]\overline{\partial}x^a\partial x^b, \end{aligned} \tag{5.7}$$

while the topological anomaly Ω reads

$$\begin{aligned} \Omega = \int_\Sigma \frac{i}{2(Z^2-16)}\big((C+5)^2-Z^2\big)PJ^a{}_b(x) \\ \times \big(\chi_a\overline{\partial}x^b - \overline{\chi}_a\partial x^b + \overline{\psi}^b\nabla y_a - \psi^b\overline{\nabla}y_a\big). \end{aligned} \tag{5.8}$$

We note that the integrand of $S_{\rm top}$ is indeed a closed form, since the tensors J_{ab}, P_{ab}, PJ_{ab} are antisymmetric, the tensors C_{ab}, Z_{ab} are symmetric and central and all are covariantly constant. Furthermore, $S_{\rm top}$ does not depend on the sign choice in the integrand, as is easy to check.

The topological anomaly Ω vanishes in a number of cases. If $(C+5\pm4)^a{}_b$ are pointwise invertible on M, we can chose

$$Z^a{}_b = \pm(C+5)^a{}_b \tag{5.9}$$

and make Ω vanish. The biKähler structures of type HK (see section 2) fall in this category, since, for these, $C^a{}_b = 0$. Alternatively, we see that Ω vanishes when $PJ^a{}_b = 0$. The biKähler structures of type P, in particular those of types K, K', (see section 2), have this property, since

$$K_0{}^a{}_b = 2PJ^a{}_b \tag{5.10}$$

and, for these, $K_0{}^a{}_b = 0$. It would be interesting to characterize the biKähler structures, if any, for which the topological anomaly Ω fails to vanish.

JHEP01(2006)041

When the target space biKähler geometry is such that the topological anomaly Ω does indeed vanish, we expect the corresponding biKähler sigma model to be a topological field theory, in analogy to what happens in Witten's A sigma model [1, 2]. In the A model, the topological correlators are independent from the world sheet complex structure and from the target manifold complex structure, but they do depend on the target manifold symplectic structure. For similar reasons, one would expect the biKähler model topological correlators to be independent from the world sheet complex structure and to depend only on a proper subset of the target space geometrical data. In section 8, we shall identify precisely this latter.

The above analysis provides strong evidence that the biKähler sigma model might indeed be a topological field theory akin to the A sigma model. One of the most basic features of topological field theories of cohomological type is that the functional measure of the associated quantum field theories localizes on the space of field configurations which are fixed point for the topological BRST charge [23]. In our case, these are the field configurations x^a, y_a, ψ^a, χ_a satisfying

$$\delta x^a = 0, \tag{5.11a}$$

$$\delta y_a = 0, \tag{5.11b}$$

$$\delta \psi^a = 0, \tag{5.11c}$$

$$\delta \chi_a = 0. \tag{5.11d}$$

From (4.4), the (5.11) are equivalent to the following set of equations

$$P^{ab}(x) y_b = 0, \tag{5.12a}$$

$$\Lambda_-(J^2 + J + 1)^a{}_b(x)\overline{\partial} x^b + \Lambda_- P^{ab}(x)\overline{\chi}_b = 0, \tag{5.12b}$$

$$\overline{\Lambda}_- J_a{}^b(x)\overline{\nabla} y_b = 0. \tag{5.12c}$$

The geometrical interpretation of these equations is not known to us, except for certain particular cases. We expect also that they may suffer some kind of disease for the biKähler structures, for which the topological anomaly Ω does not vanish (if any), and, perhaps, for an even larger class of such structures. A detailed investigation of these matters is beyond the scope of this paper. Here, we shall restrict ourselves to making a few general observations. When the endomorphism $P^a{}_b$ is pointwise invertible on M (e. g. for a type K or HK biKähler structure), eq. (5.12a) becomes equivalent to the equation

$$y_a = 0 \tag{5.13}$$

and eq. (5.12c) is identically satisfied. Eq. (5.12b) is a kind of generalized holomorphy condition for the embedding field x^a. For a type P biKähler structure, it reduces to

$$\Lambda_-(J^2 + J + 1)^a{}_b(x)\overline{\partial} x^b = 0, \tag{5.14}$$

on account of (3.2b). In particular, for a type K, K' structure, it yields a the customary notion of holomorphy

$$\Lambda_-{}^a{}_b(x)\overline{\partial} x^b = 0. \tag{5.15}$$

We conclude this section by recalling that, in the A model, the field configurations annihilated by the topological BRST charge make the gauge fermion Ψ vanish. Apparently, a similar property does not hold in general for the gauge fermion Ψ of the biKähler sigma model found above. We do not know whether this is a cohomological artifact of such Ψ or, else, it is a basic feature of the model.

6. The local cohomology of δ

In this section, we shall study some aspects of the local cohomology of δ. In view of the topological nature of the biKähler sigma model, shown above, this is an important step of our analysis, because of its relevance for the classification of topological observables and the study of the properties of their correlators.

Relations (4.4a), (4.4b) and (4.7a), (4.7b) show that the fields x^a, y_a generate a subcohomology of the δ cohomology, which we shall analyze next. For any p-vector $X^{a_1\ldots a_p}$, set

$$\mathscr{O}_X = \frac{1}{p!} X^{a_1\ldots a_p}(x) y_{a_1} \ldots y_{a_p}. \tag{6.1}$$

This is the most general local field containing only the fields x^a, y_a and no derivatives. Further, it is evident that $\mathscr{O}$ maps isomorphically the algebra of multivectors into the algebra of such local fields formed with x^a, y_a. Using (4.4a), (4.4b) and the fact that $\nabla_c P^{ab} = 0$, it is easy to show that

$$\delta \mathscr{O}_X = \mathscr{O}_{\sigma_{\mathrm{PL}} X}, \tag{6.2}$$

where $\sigma_{\mathrm{PL}} X^{a_1\ldots a_{p+1}}$ is the $p+1$-vector given by

$$\sigma_{\mathrm{PL}} X^{a_1\ldots a_{p+1}} = -(p+1) P^{[a_1|c|} \partial_c X^{a_2\ldots a_{p+1}]} + \frac{(p+1)p}{2} \partial_c P^{[a_1 a_2} X^{|c| a_3\ldots a_{p+1}]}. \tag{6.3}$$

Thus,

$$\delta \mathscr{O}_X = 0 \quad \Leftrightarrow \quad \sigma_{\mathrm{PL}} X^{a_1\ldots a_{p+1}} = 0, \tag{6.4a}$$
$$\mathscr{O}_X = \delta \mathscr{O}_Y \quad \Leftrightarrow \quad X^{a_1\ldots a_p} = \sigma_{\mathrm{PL}} Y^{a_1\ldots a_p}. \tag{6.4b}$$

The above construction has a simple geometric interpretation. Since $\nabla_c P^{ab} = 0$, P^{ab} is a Poisson 2-vector on M defining a Poisson structure P [24]. σ_{PL} is the well known associated nilpotent Poisson-Lichnerowicz operator on multivectors [24]. (6.2) shows that $\mathscr{O}$ is a cochain isomorphism of the Poisson-Lichnerowicz multivector cochain complex, $(\mathscr{V}^*(M), \sigma_{\mathrm{PL}})$, into the cochain complex of local fields formed by x^a, y_a with no derivatives, $(\mathscr{F}_{xy}, \delta)$. Thus, the cohomology of the latter, $H^*(\mathscr{F}_{xy}, \delta)$, is isomorphic to the Poisson-Lichnerowicz multivector cohomology $H^*_{\mathrm{PL}}(M, P)$.

For any p-form $\omega_{a_1\ldots a_p}$, define the p-vector

$$\#\omega^{a_1\ldots a_p} = P^{a_1 b_1} \ldots P^{a_p b_p} \omega_{b_1\ldots b_p}. \tag{6.5}$$

maps the algebra of forms into the algebra of multivectors. As is well known, # defines a cochain homomorphism of the de Rham differential form cochain complex, $(\Omega^*(M), d_{\mathrm{dR}})$,

into the Poisson-Lichnerowicz multivector cochain complex $(\mathscr{V}^*(M), \sigma_{\rm PL})$, and, thus, also a homomorphism of the de Rham cohomology, $H^*_{\rm dR}(M)$, into the Poisson-Lichnerowicz cohomology, $H^*_{\rm PL}(M,P)$, [24]. This homomorphism is an isomorphism, if P is pointwise invertible, i.e. it comes from a symplectic structure. Composing the maps $\#$ and $\mathcal{O}$, we have a homomorphism of $H^*_{\rm dR}(M)$ into $H^*(\mathscr{F}_{xy}, \delta)$, which is an isomorphism when P is pointwise invertible.

Let $X^{a_1\cdots a_p}$ be a p-vector such that $\sigma_{\rm PL}X^{a_1\cdots a_{p+1}} = 0$. Then, by (6.2), $\delta\mathcal{O}_X = 0$. Starting from $\mathcal{O}_X$, one can generate a triplet of local δ cohomology classes, by using the well known descent formalism [1, 2]. This is based on the mod d cohomology of δ, or, equivalently, on the cohomology of $\delta + d$, where d is the de Rham differential of Σ. Let us write $\mathcal{O}_X$ as $\mathcal{O}_X^{(0)}$ to emphasize the fact that it is a 0-form on Σ. We know that

$$\delta\mathcal{O}_X^{(0)} = 0. \tag{6.6}$$

We can integrate $\mathcal{O}_X^{(0)}$ on any 0-cycle Δ of Σ (that is evaluate it on a formal sum of points of Σ), yielding an object

$$\mathcal{O}_X^{(0)}(\Delta) = \oint_\Delta \mathcal{O}_X^{(0)}. \tag{6.7}$$

By (6.6), $\mathcal{O}_X^{(0)}(\Delta)$ satisfies clearly

$$\delta\mathcal{O}_X^{(0)}(\Delta) = 0 \tag{6.8}$$

and, so, defines a first local δ cohomology class. This class depends only on the homology class of the 0-cycle Δ, if $\mathcal{O}_X^{(0)}(\Delta)$ changes by a local δ exact term, when Δ changes by a 0-boundary. By Stokes' theorem, this happens provided there exists a local 1-form $\mathcal{O}_X^{(1)}$ field such that

$$d\mathcal{O}_X^{(0)} = \delta\mathcal{O}_X^{(1)}. \tag{6.9}$$

Let us assume this. We can integrate $\mathcal{O}_X^{(1)}$ on any 1-cycle Γ of Σ (roughly a formal sum of closed oriented paths on Σ), obtaining an object

$$\mathcal{O}_X^{(1)}(\Gamma) = \oint_\Gamma \mathcal{O}_X^{(1)}. \tag{6.10}$$

By (6.9) and Stokes' theorem, $\mathcal{O}_X^{(1)}(\Gamma)$ satisfies

$$\delta\mathcal{O}_X^{(1)}(\Gamma) = 0 \tag{6.11}$$

and, so, defines a second local δ cohomology class. By Stokes' theorem again, this class depends only on the homology class of the 1-cycle Γ, if $\mathcal{O}_X^{(1)}(\Gamma)$ changes by a local δ exact term, when Γ changes by a 1-boundary. This happens if there is a local 2-form $\mathcal{O}_X^{(2)}$ field such that

$$d\mathcal{O}_X^{(1)} = \delta\mathcal{O}_X^{(2)}. \tag{6.12}$$

We can integrate $\mathcal{O}_X^{(2)}$ on Σ, yielding

$$\mathcal{O}_X^{(2)}(\Sigma) = \oint_\Sigma \mathcal{O}_X^{(2)}. \tag{6.13}$$

By (6.12) and Stokes' theorem, $\mathcal{O}_X^{(2)}(\Sigma)$ satisfies

$$\delta\mathcal{O}_X^{(2)}(\Sigma) = 0 \tag{6.14}$$

and, so, defines a third δ cohomology class. Clearly, since Σ is 2-dimensional, the iterative procedure outlined above stops here. Next, let us find expressions for the descendant fields $\mathcal{O}_X^{(q)}$, $q = 0, 1, 2$.

As an ansatz, we write

$$\mathcal{O}_X^{(q)} = \frac{1}{q!(p-q)!} X^{(q)}{}_{b_1\ldots b_q}{}^{a_1\ldots a_{p-q}}(x) dx^{b_1}\ldots dx^{b_q} y_{a_1}\ldots y_{a_{p-q}}, \tag{6.15}$$

where the tensor $X^{(q)}{}_{b_1\ldots b_q}{}^{a_1\ldots a_{p-q}}$ is a q-form $p-q$-vector (i.e. it is antisymmetric in the upper and lower indices) and $q = 0, 1, 2$. Then, by (4.4a), (4.4b), the descent equations (6.6), (6.9), (6.12) hold provided

$$X^{(q-1)}{}_{b_1\ldots b_{q-1}}{}^{a_1\ldots a_{p-q+1}} - P^{a_1c} X^{(q)}{}_{b_1\ldots b_{q-1}c}{}^{a_2\ldots a_{p-q+1}} = 0, \tag{6.16}$$

for $q = 1, 2$ and

$$\sigma_{\mathrm{PL}} X^{(q)}{}_{b_1\ldots b_q}{}^{a_1\ldots a_{p-q+1}} - (-1)^q q\partial_{[b_1} X^{(q-1)}{}_{b_2\ldots b_q]}{}^{a_1\ldots a_{p-q+1}} = 0, \tag{6.17}$$

for $q = 0, 1, 2$, where, for a generic q-form p-vector $X_{b_1\ldots b_q}{}^{a_1\ldots a_p}$

$$\sigma_{\mathrm{PL}} X_{b_1\ldots b_q}{}^{a_1\ldots a_{p+1}} = -(p+1) P^{[a_1|c|}\partial_c X_{b_1\ldots b_q}{}^{a_2\ldots a_{p+1}]} \tag{6.18}$$
$$+ \frac{p(p+1)}{2}\partial_c P^{[a_1a_2} X_{b_1\ldots b_q}{}^{|c|a_3\ldots a_{p+1}]} + (-1)^q(p+1)q\partial_{[b_1} P^{[a_1|c|} X_{b_2\ldots b_q]c}{}^{a_2\ldots a_{p+1}]}.$$

Above, σ_{PL} is the generalized Poisson-Lichnerowicz operator. It acts naturally and is covariantly defined on the space of q-form p-vectors $X_{b_1\ldots b_q}{}^{a_1\ldots a_p}$ satsifying the algebraic constraint

$$P^{ab} X_{b_1\ldots b_{q-1}b}{}^{a_1\ldots a_p} = 0, \tag{6.19}$$

for $q \geq 1$. It can be shown that $\sigma_{\mathrm{PL}} X_{b_1\ldots b_q}{}^{a_1\ldots a_{p+1}}$ is a q-form $p+1$-vector fulfilling (6.19) as well and that $\sigma_{\mathrm{PL}}\sigma_{\mathrm{PL}} X_{b_1\ldots b_q}{}^{a_1\ldots a_{p+2}} = 0$. Thus, if we denote by $\mathscr{V}_q^p(M)$ the space of q-form p-vector for which (6.19) holds, we have a generalized Poisson-Lichnerowicz q-form multivector cochain complex $(\mathscr{V}_q^*(M), \sigma_{\mathrm{PL}})$, and, associated with it, a generalized Poisson-Lichnerowicz q-form multivector cohomology $H^*_{\mathrm{PL}q}(M,P)$, for every $q \geq 0$. The complex and its cohomology are trivial for $q \geq 1$, if P is pointwise invertible and, so, comes from a symplectic structure, but they are not so in general.

In (6.17), the tensor $X^{(q)}{}_{b_1\ldots b_q}{}^{a_1\ldots a_{p-q}}$ satisfies the constraint (6.16) rather than (6.19), and, therefore, $\sigma_{\mathrm{PL}} X^{(q)}{}_{b_1\ldots b_q}{}^{a_1\ldots a_{p-q+1}}$ is not covariantly defined. However, the combination of the two terms of the left hand side of (6.17) is covariant, so that (6.17) makes sense.

We do not know any general conditions ensuring the existence of solutions of the descent equations (6.16), (6.17). However, it is not difficult to show that there exists a solution when the p-vector $X^{a_1\ldots a_p}$ we start with is of the form $\#\omega^{a_1\ldots a_p}$ (cf. eq. (6.5)), for some closed p-form $\omega_{a_1\ldots a_p}$,

$$\partial_{[a_1}\omega_{a_2\ldots a_{p+1}]} = 0. \tag{6.20}$$

Indeed, in this case, the q-form p-vectors

$$X^{(q)}{}_{b_1\ldots b_q}{}^{a_1\ldots a_{p-q}} = P^{a_1c_1}\ldots P^{a_{p-q}c_{p-q}}\omega_{b_1\ldots b_q c_1\ldots c_{p-q}}, \tag{6.21}$$

$q=0,1,2$, satisfy eqs. (6.16), (6.17), as is straightforward to verify. So, when P^{ab} comes from a symplectic structure, we recover the usual de Rham descent sequence.

If a q-form $p-q$-vector $X^{(q)}{}_{b_1\ldots b_q}{}^{a_1\ldots a_{p-q}}$ satisfies (6.19), then the field $\mathcal{O}^{(q)}_X$, given by (6.15), has the property that

$$\delta\mathcal{O}^{(q)}_X = \mathcal{O}^{(q)}_{\sigma_{\mathrm{PL}}X}. \tag{6.22}$$

So, if the generalized Poisson-Lichnerowicz cohomology spaces $H^{p-q}_{\mathrm{PL}q}(M,P)$, $q=1,2$, do not vanish, the descent sequence $\mathcal{O}^{(0)}_X(\Delta)$, $\mathcal{O}^{(1)}_X(\Gamma)$, $\mathcal{O}^{(2)}_X(\Sigma)$ constructed above is not uniquely determined by $X^{a_1\ldots a_p}$ and the cycles Δ, Γ, Σ. This is, we believe, a novel feature of the biKähler model.

We remark that the cohomological setup expounded above depends on the target space biKähler geometrical data only through the combination P^{ab}. This reflects the topological nature of the associated field theory.

Of course, the above analysis does not exhaust the whole local δ cohomology. A full computation of the cohomology would be interesting, but, unfortunately, it is rather difficult because δ is nilpotent only on shell.

7. Special biKähler sigma models

In this section, we shall consider the biKähler sigma models associated with the special biKähler structures considered at the second half of section 2, since we expect these models to have special properties, which call for a closer inspection. We shall also find that one of these models is just Witten's A topological sigma model [1, 2].

We consider first a biKähler structures g_{ab}, $K_\pm{}^a{}_b$ of the closely related types K, K' and P (cf. eqs. (2.12), (2.14)). Recall that a biKähler structure of type K, K' corresponds to an ordinary Kähler structure g_{ab}, $K^a{}_b$, where $K^a{}_b$ is given by eq. (2.13). Recall also that a biKähler structure of type P induces a riemannian product structure g_{ab}, $L^a{}_b$, where $L^a{}_b$ is given by eq. (2.15). Finally, recall that a biKähler structure of type K, K' is also a particular biKähler structure of type P for which $L^a{}_b=\delta^a{}_b, -\delta^a{}_b$, respectively. Below, for all these three types of structures, we shall set

$$K^a{}_b = K_-{}^a{}_b \tag{7.1}$$

and

$$\Lambda^a{}_b = \Lambda_-{}^a{}_b \tag{7.2}$$

(cf. eqs. (2.13), (2.9)). We further set

$$Q_\pm{}^a{}_b = \frac{1}{2}(1\pm L)^a{}_b. \tag{7.3}$$

$Q_\pm{}^a{}_b$ is the orthogonal projector on the ±1 eigenbundle of $L^a{}_b$. $Q_-{}^a{}_b=0$, $Q_+{}^a{}_b=0$, when the type P structure is a type K, K' structure, respectively.

JHEP01(2006)041

From (3.2), the fields ψ^a, χ_a of the associated biKähler sigma models satisfy the constraints

$$\psi^a = \overline{\Lambda}^a{}_b(x)\psi^b, \tag{7.4a}$$

$$\chi_a = \Lambda_a{}^b(x)\chi_b, \tag{7.4b}$$

for all three types of biKähler structure.

For a type K biKähler structure, the action of the biKähler sigma model (3.3) takes the form

$$S_K = \int_\Sigma \Big\{\frac{1}{2}(ig_{ab}(x) - 3K_{ab}(x))\overline{\partial}x^a\partial x^b - ig^{ab}(x)\overline{\chi}_a\chi_b \\ + \psi^a\overline{\nabla}y_a + \overline{\psi}^a\nabla y_a - R^a{}_{ced}K^{eb}(x)\overline{\psi}^c\psi^d y_a y_b\Big\}. \tag{7.5}$$

The symmetry variations (4.4) of the fields become

$$\delta x^a = -K^{ab}(x)y_b, \tag{7.6a}$$

$$\delta y_a = \Gamma^b{}_{ad}K^{dc}(x)y_b y_c, \tag{7.6b}$$

$$\delta\psi^a = \Gamma^a{}_{bd}K^{dc}(x)\psi^b y_c + \overline{\Lambda}^a{}_b(x)\partial x^b, \tag{7.6c}$$

$$\delta\chi_a = \Gamma^b{}_{ad}K^{dc}(x)\chi_b y_c. \tag{7.6d}$$

For a type K' biKähler structure, the action (3.3) takes the simple form

$$S_{K'} = \int_\Sigma i\big(\overline{\chi}_a\partial x^a - \chi_a\overline{\partial}x^a + \overline{\psi}^a\nabla y_a - \psi^a\overline{\nabla}y_a\big). \tag{7.7}$$

The symmetry variations (4.4) of the fields become

$$\delta x^a = 0, \tag{7.8a}$$

$$\delta y_a = 0, \tag{7.8b}$$

$$\delta\psi^a = -i\overline{\Lambda}^a{}_b(x)\partial x^b, \tag{7.8c}$$

$$\delta\chi_a = -i\Lambda_a{}^b(x)\nabla y_b. \tag{7.8d}$$

For a type P biKähler structure, the action (3.3) reads as

$$S_P = \int_\Sigma \Big\{\frac{1}{2}(iQ_{+ab}(x) - 3Q_+K_{ab}(x))\overline{\partial}x^a\partial x^b - iQ_+{}^{ab}(x)\overline{\chi}_a\chi_b \\ + iQ_-{}^a{}_b(x)(\overline{\chi}_a\partial x^b - \chi_a\overline{\partial}x^b) + Q_+{}^a{}_b(x)(\psi^b\overline{\nabla}y_a + \overline{\psi}^b\nabla y_a) \\ + iQ_-{}^a{}_b(x)(\overline{\psi}^b\nabla y_a - \psi^b\overline{\nabla}y_a) - R^a{}_{ced}Q_+K^{eb}(x)\overline{\psi}^c\psi^d y_a y_b\Big\}. \tag{7.9}$$

The symmetry variations (4.4) of the fields become

$$\delta x^a = -Q_+K^{ab}(x)y_b, \tag{7.10a}$$

$$\delta y_a = \Gamma^b{}_{ad}Q_+K^{dc}(x)y_b y_c, \tag{7.10b}$$

$$\delta\psi^a = \Gamma^a{}_{bd}Q_+K^{dc}(x)\psi^b y_c + \overline{\Lambda}(Q_+ - iQ_-)^a{}_b(x)\partial x^b, \tag{7.10c}$$

$$\delta\chi_a = \Gamma^b{}_{ad} Q_+ K^{dc}(x)\chi_b y_c - i\Lambda Q_{-a}{}^b(x)\nabla y_b. \tag{7.10d}$$

It is remarkable that, for the K, K' and P models, the gauge fermion Ψ and the topological action $S_{\rm top}$ entering the basic relation (5.1) (cf. eqs. (5.3), (5.5), (5.6), (5.7)) do not depend on the choice of the central element $Z^a{}_b$. The gauge fermion is given by the same expression for the three models

$$\Psi_K = \Psi_{K'} = \Psi_P = \int_\Sigma \frac{i}{2} g_{ab}(x)(\overline{\psi}^a \partial x^b - \psi^a \overline{\partial} x^b). \tag{7.11}$$

The topological term is given by

$$S_{{\rm top}K} = -\int_\Sigma K_{ab}(x)\overline{\partial} x^a \partial x^b, \tag{7.12}$$

for the K model

$$S_{{\rm top}K'} = 0, \tag{7.13}$$

for the K' model, and, finally

$$S_{{\rm top}P} = -\int_\Sigma Q_+ K_{ab}(x)\overline{\partial} x^a \partial x^b, \tag{7.14}$$

for the P model.

Using the fixed point theorem of ref. [23], or directly from eqs. (5.12), we can write down with little effort the equations defining the moduli space associated with the K, K' and P models. For the K model, they read

$$\Lambda^a{}_b(x)\overline{\partial} x^b = 0, \tag{7.15a}$$

$$y_b = 0. \tag{7.15b}$$

For the K' model, we find

$$\Lambda^a{}_b(x)\overline{\partial} x^b = 0, \tag{7.16a}$$

$$\overline{\Lambda}_a{}^b(x)\overline{\nabla} y_b = 0. \tag{7.16b}$$

Finally, for the P model, we have

$$Q_+ K^{ab}(x) y_b = 0, \tag{7.17a}$$

$$\Lambda(Q_+ + iQ_-)^a{}_b(x)\overline{\partial} x^b = 0, \tag{7.17b}$$

$$\overline{\Lambda} Q_{-a}{}^b(x)\overline{\nabla} y_b = 0. \tag{7.17c}$$

Let us discuss the above results. In the K model, the field χ_a is non propagating and decouples from the rest of the fields in the action, a peculiarity of the K model, which is not shared by the other biKähler sigma models. Thus, χ_a may be set to zero

$$\chi_a = 0. \tag{7.18}$$

This is consistent with the variation (7.6d). After this is done, by inspection of (7.5), (7.6), one realizes immediately that K model is nothing but Witten's A topological sigma model

[1, 2] up to a few minor differences. In the usual formulation of the A model, instead of the field y_a, one uses the related field

$$r^a = -K^{ab}(x)y_b. \tag{7.19}$$

Its symmetry variation is

$$\delta r^a = 0. \tag{7.20}$$

Further, the normalization of the topological term is different from the one given here. From (7.15), we see that the moduli space of the K model classifies holomorphic embeddings x^a of the world sheet Σ into the target manifold M, in agreement with the well known property of the A model.

By inspecting (7.7), it is easy to see that the K' model is nothing but the infinite radius limit of the A model (in the first order formulism) [1] (see also [25]). So, we may call it also the A' model. This is a sort of bc system, though, strictly speaking, it is not, since the operator ∇ contains a non linear x^a dependence via the connection coefficients $\Gamma^a{}_{bc}$. The A' model has another symmetry besides (7.8). However, this holds only in the infinite radius limit, in which non invariance terms proportional to g^{ab} can be neglected. (7.8), conversely, is an exact symmetry. From (7.16), it appears the moduli space of the K' model classifies pairs (x^a, y_a) constituted by a holomorphic embedding of Σ into M and a holomorphic section of $x^*\Pi T^{*1,0}M$.

The P model interpolates between the K and K' model, to which it reduces, when $L^a{}_b = \delta^a{}_b, \ -\delta^a{}_b$, respectively. The P model, as far as we know, is not related to any known topological sigma model. Also its moduli space is apparently unknown.

Consider next a biKähler structure of type HK corresponding to a hyperKähler structure g_{ab}, $K_i{}^a{}_b$ (cf. eqs. (2.16), (2.17)). Below, we set

$$\Lambda_2{}^a{}_b = \Lambda_-{}^a{}_b. \tag{7.21}$$

(cf. eq. (2.9)).

For a biKähler type HK target structure, the fields ψ^a, χ_a satisfy

$$\psi^a = \overline{\Lambda}_2{}^a{}_b(x)\psi^b, \tag{7.22a}$$

$$\chi_a = \Lambda_{2a}{}^b(x)\chi_b. \tag{7.22b}$$

For a type HK biKähler structure, the action (3.3) reads as

$$\begin{aligned} S_{HK} = \int_\Sigma \Big\{ & \frac{1}{4}(ig_{ab}(x) - 5K_{2ab}(x))\overline{\partial}x^a\partial x^b - ig^{ab}(x)\overline{\chi}_a\chi_b \\ & + S^{ab}(x)\big(\overline{\chi}_a + g_{ac}(x)\overline{\partial}x^c\big)\big(\chi_b + g_{bd}(x)\partial x^d\big) \\ & + \frac{1}{2}(1+2S)^a{}_b(x)\big(\psi^b\overline{\nabla}y_a + \overline{\psi}^b\nabla y_a\big) + R^a{}_{ced}P^{eb}(x)\overline{\psi}^c\psi^d y_a y_b\Big\}, \end{aligned} \tag{7.23}$$

where the tensors $P^a{}_b$, $S^a{}_b$ are given by

$$P^a{}_b = \frac{1}{2}(K_1 - K_2)^a{}_b, \tag{7.24a}$$

$$S^a{}_b = \frac{1}{2}(K_1 + K_2 + K_3)^a{}_b \tag{7.24b}$$

The symmetry variations (4.4) of the fields become

$$\delta x^a = P^{ab}(x) y_b, \tag{7.25a}$$

$$\delta y_a = -\Gamma^b{}_{ad} P^{dc}(x) y_b y_c, \tag{7.25b}$$

$$\delta \psi^a = -\Gamma^a{}_{bd} P^{dc}(x) \psi^b y_c + \Big(\frac{1}{2}\overline{\Lambda}_2{}^a{}_b(x) + F_-{}^a{}_b(x)\Big)\partial x^b + G_-{}^{ab}(x)\chi_b, \tag{7.25c}$$

$$\delta \chi_a = -\Gamma^b{}_{ad} P^{dc}(x) \chi_b y_c + F_+{}^b{}_a(x) \nabla y_b - R^b{}_d{}^c{}_e G_+{}^e{}_a(x) \psi^d y_b y_c. \tag{7.25d}$$

where the tensors $F_\pm{}^a{}_b$, $G_\pm{}^a{}_b$ are given by

$$F_\pm{}^a{}_b = \frac{1}{4}(K_1 + K_2 - i \pm i K_3)^a{}_b, \tag{7.26a}$$

$$G_\pm{}^a{}_b = \frac{1}{4}(K_1 - K_2 + i \pm i K_3)^a{}_b. \tag{7.26b}$$

For the HK sigma model, the gauge fermion Ψ and the topological action $S_{\rm top}$ entering relation (5.1) (cf. eqs. (5.3), (5.5), (5.6), (5.7)) are given by

$$\Psi_{HK} = \int_\Sigma \frac{i}{2}\Big[\big(g_{ab}(x) + H_1 K_{3ab}(x)\big)(\overline{\psi}^a \partial x^b - \psi^a \overline{\partial} x^b) + H_2{}^a{}_b(x)(\overline{\chi}_a \psi^b - \chi_a \overline{\psi}^b)\Big], \tag{7.27}$$

where the tensors $H_1{}^a{}_b$, $H_2{}^a{}_b$ are given by

$$H_1{}^a{}_b = \frac{1}{18}(K_1 - 17K_2 - 4K_3)^a{}_b, \qquad H_2{}^a{}_b = \frac{1}{18}(K_1 + K_2 + 5K_3)^a{}_b, \tag{7.28}$$

and

$$S_{{\rm top}HK} = \int_\Sigma 2P_{ab}(x)\overline{\partial} x^a \partial x^b. \tag{7.29}$$

The expression of Ψ_{HK} is not particularly illuminating, though its complexity may be a cohomological artifact.

The moduli space of the HK model is defined by the equations

$$\Big(\frac{1}{2}\Lambda_2{}^a{}_b(x) + \overline{F}_-{}^a{}_b(x)\Big)\overline{\partial} x^b + \overline{G}_-{}^{ab}(x)\overline{\chi}_b = 0, \tag{7.30a}$$

$$y_a = 0 \tag{7.30b}$$

obtainable for instance from (5.12), (5.13). These equations are characterized by the explicit appearance of the field χ_a, in contrast to what happens for the K, K' and P models.

To the best of our knowledge, the HK model is not related to any known topological sigma model. Also the associated moduli space is apparently unknown. The fact that the field χ_a appears explicitly in the moduli space equations (7.30) may indicate that it plays a role rather different from that it does in the K, K' and P models. At this stage, it is difficult to assess the relevance and even the consistency of the HK model.

JHEP01(2006)041

8. Relation to the Hitchin model

In this section, we review briefly the Hitchin sigma model, worked out in refs. [19, 20], restricting ourselves to the case where the target space generalized complex structure is actually a generalized Kähler structure [8, 9]. We then show that the action and the symmetries of the biKähler sigma model can be obtained by gauge fixing the Batalin-Vilkovisky master action of the Hitchin model by restricting to a suitably chosen submanifold of field space, that is lagrangian with respect to the Batalin-Vilkovisky odd symplectic form.

In general, the fields of a 2-dimensional field theory are differential forms on a oriented 2-dimensional manifold Σ. They can be viewed as elements of the space $\mathrm{Fun}(\Pi T\Sigma)$ of functions on the parity reversed tangent bundle $\Pi T\Sigma$ of Σ, which we shall call de Rham superfields. More explicitly, we associate with the coordinates t^α of Σ Grassmann odd partners τ^α with $\deg t^\alpha = 0$, $\deg \tau^\alpha = 1$. A de Rham superfield $\underline{\psi}(t,\tau)$ is a triplet formed by a 0–, 1–, 2-form field $\psi^{(0)}(t)$, $\psi^{(1)}{}_\alpha(t)$, $\psi^{(2)}{}_{\alpha\beta}(t)$ organized as

$$\underline{\psi}(t,\tau) = \psi^{(0)}(t) + \tau^\alpha \psi^{(1)}{}_\alpha(t) + \frac{1}{2}\tau^\alpha\tau^\beta\psi^{(2)}{}_{\alpha\beta}(t). \tag{8.1}$$

The forms $\psi^{(0)}$, $\psi^{(1)}$, $\psi^{(2)}$ are called the de Rham components of $\underline{\psi}$.

$\Pi T\Sigma$ is endowed with a natural differential $\underline{d}$ defined by

$$\underline{d}t^\alpha = \tau^\alpha, \qquad \underline{d}\tau^\alpha = 0. \tag{8.2}$$

In this way, the exterior differential d of Σ can be identified with the operator

$$\underline{d} = \tau^\alpha \partial_\alpha. \tag{8.3}$$

The coordinate invariant integration measure of $\Pi T\Sigma$ is

$$\mu = \mathrm{d}t^1 \mathrm{d}t^2 \mathrm{d}\tau^1 \mathrm{d}\tau^2. \tag{8.4}$$

Any de Rham superfield $\underline{\psi}$ can be integrated on $\Pi T\Sigma$ according to the prescription

$$\int_{\Pi T\Sigma} \mu\, \underline{\psi} = \int_\Sigma \frac{1}{2}\mathrm{d}t^\alpha \mathrm{d}t^\beta \psi^{(2)}{}_{\alpha\beta}(t). \tag{8.5}$$

The components of the relevant de Rham superfields carry, besides the form degree, also a ghost degree. We shall limit ourselves to homogeneous superfields, that is superfields $\underline{\psi}$ for which the sum of the form and ghost degree is the same for the three components $\psi^{(0)}$, $\psi^{(1)}$, $\psi^{(2)}$ of $\underline{\psi}$. The common value of that sum is the superfield (total) degree $\deg \underline{\psi}$. It is easy to see that the differential operator $\underline{d}$ and the integration operator $\int_{\Pi T\Sigma} \mu$ carry degree 1 and -2, respectively.

It is often necessary to choose a complex structure on Σ. With this, there are associated complex coordinates for Σ, z, and their Grassmann odd partners, ζ, and their complex conjugates. As before, $\deg z = 0$, $\deg \zeta = 1$. All the above relations can be written in terms these coordinates, if one wishes so. Further, once a complex structure is given, we can define the Cauchy-Riemann operator

$$\underline{\partial} = \zeta \partial_z \tag{8.6}$$

and its complex conjugate and, with this, a notion of holomorphy for superfields. $\underline{\partial}$ has obviously degree 1.

Now, we shall introduce the Hitchin sigma model [19, 20]. The basic fields of the model are a degree 0 superembedding $\underline{x} \in \Gamma(\Pi T\Sigma, M)$ and a degree 1 supersection $\underline{y} \in \Gamma(\Pi T\Sigma, x^*\Pi T^*M)$, where ΠT^*M is the parity reversed cotangent bundle of M. With respect to each local coordinate of M, $\underline{x}$, $\underline{y}$ are given as de Rham superfields $\underline{x}^a$, $\underline{y}_a$. The Batalin-Vilkovisky odd symplectic form is

$$\Omega_{BV} = \int_{\Pi T\Sigma} \mu\, \delta\underline{x}^a \delta\underline{y}_a, \tag{8.7}$$

where, here, δ denotes the differential operator in field space. Ω_{BV} is a closed functional form, $\delta\Omega_{BV} = 0$. In this way, one can define Batalin-Vilkovisky antibrackets $(\,,)_{BV}$ in standard fashion by the formula:

$$(F,G)_{BV} = \int_{\Pi T\Sigma} \mu \left[\frac{\delta_r F}{\delta\underline{x}^a}\frac{\delta_l G}{\delta\underline{y}_a} - \frac{\delta_r F}{\delta\underline{y}_a}\frac{\delta_l G}{\delta\underline{x}^a}\right], \tag{8.8}$$

for any two functionals F, G of $\underline{x}^a$, $\underline{y}_a$, where the subscripts l, r denote left, right functional differentiation, respectively.

In the Hitchin sigma model, the target space geometry is specified by a generalized complex structure $\mathcal{J}^A{}_B$ [8]. In the case under our study, the structure $\mathcal{J}^A{}_B$ is the generalized Kähler structure corresponding to a biKähler structure g_{ab}, $K_\pm{}^a{}_b$ [9] and, in block form, is given by

$$\mathcal{J}^A{}_B = \begin{pmatrix} J^a{}_b & P^{ab} \\ P_{ab} & J_a{}^b \end{pmatrix}, \tag{8.9}$$

where the tensors $J^a{}_b$, $P^a{}_b$ are given by (3.4a). The action of the associated Hitchin model is

$$S_{GK} = \int_{\Pi T\Sigma} \mu \left[\frac{1}{2}P^{ab}(\underline{x})\underline{y}_a\underline{y}_b + J^a{}_b(\underline{x})\underline{y}_a\underline{dx}^b + \frac{1}{2}P_{ab}(\underline{x})\underline{dx}^a\underline{dx}^b\right]. \tag{8.10}$$

Actually, in the Hitchin sigma model, as formulated in [19, 20], the action S_{GK} contains an extra term $\int_{\Pi T\Sigma} \mu\, \underline{y}_a\underline{dx}^a$ absent here. It is straightforward to see that this omission does not alter the main property of the model, that is the correspondence between the integrability conditions of the generalized almost complex structure $\mathcal{J}^{\mathcal{A}}{}_{\mathcal{B}}$ and the restrictions on target space geometry implied by the Batalin-Vilkovisky classical master equation. Indeed, it can be checked that S_{GK} satisfies the classical Batalin-Vilkovisky master equation

$$(S_{GK}, S_{GK})_{BV} = 0, \tag{8.11}$$

as a consequence of (2.1), (2.2), (2.3).

The Batalin-Vilkovisky variations $\delta_{BV}\underline{x}^a$, $\delta_{BV}\underline{y}_a$ are defined by

$$\delta_{BV}\underline{x}^a = (S_{GK}, \underline{x}^a)_{BV}, \tag{8.12a}$$

$$\delta_{BV}\underline{y}_a = (S_{GK}, \underline{y}_a)_{BV}. \tag{8.12b}$$

Using (8.10), (8.12), it is straightforward to obtain the explicit expressions of $\delta_{BV}\underline{x}^a$, $\delta_{BV}\underline{y}_a$

$$\delta_{BV}\underline{x}^a = P^{ab}(\underline{x})\underline{y}_b + J^a{}_b(\underline{x})\underline{dx}^b, \tag{8.13a}$$

$$\delta_{BV}\underline{y}_a = -\Gamma^b{}_{ad}P^{dc}(\underline{x})\underline{y}_b\underline{y}_c - \Gamma^b{}_{ad}J^d{}_c(\underline{x})\underline{y}_b\underline{dx}^c + J^b{}_a(\underline{x})\underline{\nabla y}_b, \tag{8.13b}$$

where

$$\underline{\nabla} = \underline{d} \pm \Gamma^{\cdot}{}_{\cdot a}(\underline{x})\underline{dx}^a. \tag{8.14}$$

The operator δ_{BV} has degree $+1$. As is well known, the master equation (8.11) implies that δ_{BV} is nilpotent

$${\delta_{BV}}^2 = 0. \tag{8.15}$$

The associated cohomology is the classical Batalin-Vilkovisky cohomology. Also, by (8.11), one has

$$\delta_{BV}S_{GK} = 0. \tag{8.16}$$

As it is, the action S_{GK} is not suitable for quantization because it possesses a gauge symmetry as a consequence of (8.16). This gauge symmetry renders the kinetic terms of the fields ill defined. Gauge fixing is required. (We refer the reader to ref. [26] for an exhaustive treatment of gauge fixing in the framework of the Batalin-Vilkovisky quantization algorithm.) As is well known, this is carried out by restricting the action to a field space submanifold $\mathfrak{L}$, that is lagrangian with respect to the Batalin-Vilkovisky odd symplectic form Ω_{BV}. The resulting quantum field theory does not depend on the choice of $\mathfrak{L}$ for continuous deformation of the latter. However, not every choice of $\mathfrak{L}$ leads to a well defined quantum field theory. A particular choice of $\mathfrak{L}$, then, can be justified only *a posteriori*. Below, we shall implement the gauge fixing following closely the methodology of Alexandrov, Kontsevich, Schwartz and Zaboronsky [22], with which they worked out a formulation of the A topological sigma model *à la* Batalin-Vilkovisky.

The definition of $\mathfrak{L}$ requires the choice of a complex structure on Σ. With this given, we define the differential operator

$$\underline{D} = \zeta\partial_\zeta \pm \Gamma^{\cdot}{}_{\cdot a}(\underline{x})\zeta\partial_\zeta\underline{x}^a \tag{8.17}$$

and its complex conjugate, of degree 0. $\underline{D}$ turns out to be very useful because of its remarkable properties. $\underline{D}$ is a projector, as

$$\underline{D}^2 = \underline{D}, \tag{8.18}$$

as is easy to check. Further, one has

$$\int_{\Pi T\Sigma} \mu\,\underline{\psi} = \int_{\Pi T\Sigma} \mu\,\underline{\overline{D}D\psi}, \tag{8.19}$$

for any superfield $\underline{\psi}$ of the our sigma model.

The field space submanifold $\mathfrak{L}$ is defined by the constraints

$$\Lambda_-{}^a{}_b(\underline{x})\underline{Dx}^b \simeq 0, \tag{8.20a}$$

$$\overline{\Lambda}_{-a}{}^{b}(\underline{x})\underline{D}\big[\underline{y}_b + ig_{bc}(\underline{x})(\underline{\partial x}^c + \underline{\overline{\partial} x}^c)\big] \simeq 0, \tag{8.20b}$$

where $\Lambda_-{}^a{}_b$ is the projector (2.9). Here and below, the symbol $\simeq$ denotes equality holding upon restriction to $\mathfrak{L}$ in field space. By direct verification, one can show that the Batalin-Vilkovisky odd symplectic form vanishes on $\mathfrak{L}$,

$$\Omega_{BV} \simeq 0. \tag{8.21}$$

In more precise terms, it is the the pull-back of Ω_{BV} by the injection $\iota_{\mathfrak{L}}$ of $\mathfrak{L}$ into field space, $\iota_{\mathfrak{L}}{}^*\Omega_{BV}$, that vanishes. Thus, $\mathfrak{L}$ is a field space lagrangian submanifold for the Batalin-Vilkovisky symplectic form Ω_{BV}, as desired. The verification of (8.21) is straightforward though lengthy. Here, we shall provide a few usefull hints about the way it is carried out. To controll covariance, one rewrites (8.7) conveniently as

$$\Omega_{BV} = \int_{\Pi T\Sigma} \mu\, \delta\underline{x}^a \delta_{\mathrm{c}}\underline{y}_a, \tag{8.22}$$

where the covariant variations $\delta_{\mathrm{c}}\underline{y}_a$ is given by

$$\delta_{\mathrm{c}}\underline{y}_a = \delta\underline{y}_a - \Gamma^b{}_{ac}(\underline{x})\delta\underline{x}^c\underline{y}_b. \tag{8.23}$$

Using (8.19), one has then

$$\Omega_{BV} = \int_{\Pi T\Sigma} \mu\, \big[\underline{\overline{D}D}\delta\underline{x}^a\delta_{\mathrm{c}}\underline{y}_a + \underline{\overline{D}}\delta\underline{x}^a\underline{D}\delta_{\mathrm{c}}\underline{y}_a + \underline{D}\delta\underline{x}^a\underline{\overline{D}}\delta_{\mathrm{c}}\underline{y}_a + \delta\underline{x}^a\underline{\overline{D}D}\delta_{\mathrm{c}}\underline{y}_a\big]. \tag{8.24}$$

By applying the operator δ to the constraints (8.20), one obtains relations involving $\underline{D}\delta\underline{x}^a$, $\underline{D}\delta_{\mathrm{c}}\underline{y}_a$, $\underline{\overline{D}D}\delta\underline{x}^a$, $\underline{\overline{D}D}\delta_{\mathrm{c}}\underline{y}_a$ and their complex conjugates, which, together with (8.20), allow one to show (8.21).

Using (8.19), it is simple to show that

$$\begin{aligned} S_{GK} = \int_{\Pi T\Sigma} \mu\, \Big[& P^{ab}(\underline{x})\big(\underline{\overline{D}y}_a\underline{Dy}_b + \underline{y}_a\underline{\overline{D}Dy}_b\big) + P_{ab}(\underline{x})\underline{\overline{\partial}x}^a\underline{\partial x}^b \\ & + J^a{}_b(\underline{x})\big(\underline{Dy}_a\underline{\overline{\partial}x}^b + \underline{\overline{D}y}_a\underline{\partial x}^b + y_a(\underline{\overline{\nabla}Dx}^b + \underline{\nabla\overline{D}x}^b)\big)\Big], \end{aligned} \tag{8.25}$$

where $\underline{\nabla}$ is the covariant Cauchy-Riemann operator

$$\underline{\nabla} = \underline{\partial} \pm \Gamma^{\cdot}{}_{\cdot a}(\underline{x})\underline{\partial x}^a \tag{8.26}$$

and similarly for its complex conjugate.[3] Let us call a superfield $\underline{\phi}$ of one of the forms $\underline{\phi} = \underline{D\psi}$, $\underline{\phi} = \underline{\overline{D}\psi}$, $\underline{\phi} = \underline{\overline{D}D\psi}$ a descendent of $\underline{\psi}$. (8.25) shows that S_{GK} is a functional of the superfields $\underline{x}^a$, $\underline{y}_a$ both explicitly and implicitly through their descendent superfields $\underline{Dx}^a$, $\underline{Dy}_a$ and $\underline{\overline{D}Dy}_a$ and their complex conjugates. On the lagrangian submanifold $\mathfrak{L}$, the superfields $\underline{Dx}^a$, $\underline{Dy}_a$ and $\underline{\overline{D}Dy}_a$ satisfy certain relations entailed by (8.20). A detailed analysis shows that these relations allow one to express $\underline{Dx}^a$, $\underline{Dy}_a$ and $\underline{\overline{D}Dy}_a$ in terms of the superfields $\underline{x}^a$, $\underline{y}_a$ and the further superfields

$$\underline{\psi}^a = \overline{\Lambda}_-{}^a{}_b(\underline{x})\underline{Dx}^b, \tag{8.27a}$$

[3] Here, we are changing our notation with respect to (8.14).

$$\underline{\chi}_a = \Lambda_{-a}{}^b(\underline{x})\underline{D}\underline{y}_b, \tag{8.27b}$$

explicitly, i.e. without any appearance of their descendent superfields, such as $\underline{D}\psi^a$, $\underline{D}\underline{\chi}_a$, etc.:

$$\underline{D}\underline{x}^a \simeq \underline{\psi}^a, \tag{8.28a}$$

$$\underline{D}\underline{y}_a \simeq \underline{\chi}_a - i\overline{\Lambda}_{-ab}(\underline{x})(\underline{\partial x}^b + \overline{\nabla}\underline{\psi}^b), \tag{8.28b}$$

$$\overline{\underline{D}}\underline{D}\underline{y}_a \simeq -R^b{}_{ecd}\overline{\Lambda}_-{}^e{}_a(\underline{x})\overline{\underline{\psi}}^c\underline{\psi}^d\underline{y}_b + i(\Lambda_- - \overline{\Lambda}_-)_{ab}(\underline{x})(\overline{\nabla}\underline{\psi}^b + \nabla\overline{\underline{\psi}}^b). \tag{8.28c}$$

By (8.25), we have therefore,

$$S_{GK} \simeq S_{GK}^{\mathrm{gf}}, \tag{8.29}$$

where S_{GK}^{gf} is an explicit functional of the superfields $\underline{x}^a$, $\underline{y}_a$, $\underline{\psi}^a$, $\underline{\chi}_a$. From (8.25), it appears that S_{GK}^{gf} depends only on the lowest non zero de Rham components of the superfields $\underline{x}^a$, $\underline{y}_a$, $\underline{\psi}^a$, $\underline{\chi}_a$, which we denote x^a, y_a, ψ^a, χ_a. These are precisely the fields of the biKähler sigma model introduced in section 3. From (8.27), it is evident that ψ^a, χ_a obey the constraints (3.2). Through a detailed calculation, one finds further that, in terms of x^a, y_a, ψ^a, χ_a, S_{GK}^{gf} equals the biKähler sigma model action S given in eq. (3.3). Similarly, one can derive from the Batalin-Vilkovisky variations (8.13) the symmetry variations (4.4). In this way, we were able to show the relation of the biKähler sigma model to the Hitchin sigma model for generalized Kähler target.

The result we have just obtained is interesting in itself, but it is also interesting because of the light it sheds on the nature of world sheet and target space geometrical data, on which the quantum field theory associated with biKähler sigma effectively depends. We obtained the biKähler sigma model by gauge fixing the Hitchin sigma model with generalized Kähler target following the general prescriptions of Batalin-Vilkovisky formalism. We know in this way that the resulting gauge fixed field theory depends generically on the geometrical data contained in the Hitchin sigma model action S_{GK}, but it is independent from those defining the lagrangian submanifold $\mathfrak{L}$ [22]. Now, the action S_{GK} has the following structure

$$S_{GK} = S_{GK1} + S_{GK2}, \tag{8.30}$$

where S_{GK1}, S_{GK2} are given by

$$S_{GK1} = \int_{\Pi T\Sigma} \mu \left[\frac{1}{2}P^{ab}(\underline{x})\underline{y}_a\underline{y}_b + J^a{}_b(\underline{x})\underline{y}_a\underline{dx}^b\right], \tag{8.31a}$$

$$S_{GK2} = \int_{\Pi T\Sigma} \mu\, \frac{1}{2}P_{ab}(\underline{x})\underline{dx}^a\underline{dx}^b. \tag{8.31b}$$

Since P_{ab} is a closed 2-form, S_{GK2} is just a topological term. As we remarked in section 3, terms of this type do not affect the field equations and are invariant under all infinitesimal symmetries. Their values characterize the topological sectors of the field theory, but such terms in themselves do not affect in any way the quantum structure of the field theory. Thus, they may be adjusted as one wishes as a matter of definition of the model without really changing its quantum properties in any essential way. The truly quantum sector of the field theory stems, upon gauge fixing, from S_{GK1}. This depends only on the combinations

$J^a{}_b$, P^{ab} (cf. eq. (3.4a)) of the target space geometrical data g_{ab}, $K_\pm{}^a{}_b$. The lagrangian submanifold $\mathfrak{L}$ depends also on the chosen complex structure of Σ and on g_{ab}, $K_-{}^a{}_b$, but, for the reasons recalled above, the gauge fixed field theory will not. So, we conclude that the quantum field theory associated to the biKähler model depends effectively only on the combinations $J^a{}_b$, P^{ab} of target space biKähler geometrical data. This solves the problem posed in section 5. For the A model, $J^a{}_b = 0$, $P^{ab} = -K^{ab}$. We recover in this way the well known result that the A model depends only on the target space Kähler structure.

In his thesis [9], Gualtieri showed that a biKähler geometry g_{ab}, $K_\pm{}^a{}_b$ is fully equivalent to a pair of commuting generalized complex structures

$$\mathcal{J}_1{}^A{}_B = \begin{pmatrix} J^a{}_b & P^{ab} \\ P_{ab} & J_a{}^b \end{pmatrix}, \qquad \mathcal{J}_2{}^A{}_B = \begin{pmatrix} P^a{}_b & J^{ab} \\ J_{ab} & P_a{}^b \end{pmatrix}, \tag{8.32}$$

whose product $-\mathcal{J}_1{}^A{}_C\mathcal{J}_2{}^C{}_B$ satisfies a certain positivity condition. The structure $\mathcal{J}_1{}^A{}_B$ equals the structure $\mathcal{J}^A{}_B$ of eq. (8.9) in terms of which the Hitchin model action S_{GK} is defined. So, we could reformulate the above results saying that the quantum field theory of the biKähler model depends only on $\mathcal{J}_1{}^A{}_B$, a fact implicit in the results previously obtained by several authors [17, 10, 11, 21].

9. Discussion

In this final section, we discuss our results by comparing them with those of other studies, which have appeared in the literature, and by listing the open problems.

Twisting of the (2,2) supersymmetric biKähler sigma model. We recall [3] that, for any target space M with biKähler structure g_{ab}, $K_\pm{}^a{}_b$, there exists a $(2,2)$ supersymmetric sigma model. Its action S may contain a closed 2-form b_{ab}, defining a topological term. Explicitly, in $(1,1)$ superspace notation, S is given by

$$S \;=\; \frac{1}{2}\int d^2\sigma d^2\theta\big(g_{ab}(X) + b_{ab}(X)\big)D_+X^aD_-X^b, \tag{9.1}$$

where X^a is the $(1,1)$ superfield

$$X^a = x^a + \theta^+\psi^a_+ + \theta^-\psi^a_- + \theta^-\theta^+F^a, \tag{9.2}$$

defining a superembedding of the the $(1,1)$ superworldsheet into M, and the $D_\pm$ are the supercovariant derivatives

$$D_\pm = \frac{\partial}{\partial\theta^\pm} + i\theta^\pm\partial_\pm, \qquad \partial_\pm \equiv \partial_0 \pm \partial_1. \tag{9.3}$$

The complex structures $K_\pm{}^a{}_b$ do not appear in the action, but they enter into the definition of the $(2,2)$ supersymmetry variations:

$$\delta X^a = \epsilon^+Q_+X^a + \epsilon^-Q_-X^a + \tilde\epsilon^+K_+(X)^a{}_b\tilde{Q}_+X^b + \tilde\epsilon^-K_-(X)^a{}_b\tilde{Q}_-X^b, \tag{9.4}$$

where the $\epsilon^{\pm}$, $\tilde{\epsilon}^{\pm}$ are anticommuting parameters and

$$Q_{\pm} = \frac{\partial}{\partial \theta^{\pm}} - i\theta^{\pm}\partial_{\pm}, \qquad \tilde{Q}_{\pm} = \frac{\partial}{\partial \theta^{\pm}} + i\theta^{\pm}\partial_{\pm}. \tag{9.5}$$

In refs. [10, 11], Kapustin and Li proposed a twisting prescription to generate a generalized A topological sigma model from the $(2,2)$ sigma model. After the topological twist, the fields

$$r_{+}{}^{a} = \overline{\Lambda}_{+}{}^{a}{}_{b}(x)\psi_{+}{}^{b}, \qquad r_{+}{}^{a} = \Lambda_{-}{}^{a}{}_{b}(x)\psi_{-}{}^{b} \tag{9.6}$$

become 0-form sections of $x^{*}\Pi T_{+}^{0,1}M$, $x^{*}\Pi T_{-}^{1,0}M$, while the fields

$$\phi_{+}{}^{a} = \Lambda_{+}{}^{a}{}_{b}(x)\psi_{+}{}^{b}, \qquad \phi_{-}{}^{a} = \overline{\Lambda}_{-}{}^{a}{}_{b}(x)\psi_{-}{}^{b} \tag{9.7}$$

become $(0,1)$-, $(1,0)$-form sections of $x^{*}\Pi T_{+}^{1,0}M$, $x^{*}\Pi T_{-}^{0,1}M$, respectively.[4] This prescription is rather natural, since, as (9.4) shows, the complex structures $K_{\pm}{}^{a}{}_{b}$ correspond to the two world sheet chiralities. It is clear that this field content cannot match the one of the biKähler model studied in this paper (cf. section 3). Further, according to the same authors, the topological variations of the fields x^{a}, $r_{\pm}{}^{a}$ are given by

$$\delta x^{a} = r_{+}{}^{a} + r_{-}{}^{a}, \tag{9.8a}$$

$$\delta r_{\pm}{}^{a} = -\Gamma^{a}{}_{bc}(x) r_{\mp}{}^{b} r_{\pm}{}^{c}, \tag{9.8b}$$

with

$$\delta^{2} = 0. \tag{9.9}$$

The local observables of the theory are thus of the form

$$\widehat{f} = \sum_{p,q\geq 0} \frac{1}{p!q!} f_{a_1\cdots a_p;b_1\cdots b_q}(x) r_{+}{}^{a_1}\cdots r_{+}{}^{a_p} r_{-}{}^{b_1}\cdots r_{-}{}^{b_q}, \tag{9.10}$$

where the $f_{a_1\cdots a_p;b_1\cdots b_q}$ belong to $\Omega_{+}^{0,p}(M)\otimes\Omega_{-}^{q,0}(M)$ and satisfy

$$p\overline{\Lambda}_{+}{}^{c}{}_{[a_1}\nabla_{|c|} f_{a_2\cdots a_p];b_1\cdots b_q} + (-1)^{p} q \Lambda_{-}{}^{c}{}_{[b_1}\nabla_{|c} f_{a_1\cdots a_p|;b_2\cdots b_q]} = 0. \tag{9.11}$$

The associated cohomology has no apparent relation with the Poisson-Licnerowicz cohomology found in section 6.[5]

So, seemingly, the construction of Kapustin and Li and ours are fundamentally different. As a consequence, our biKähler sigma model has no immediate interpretation as a twisted form of a $(2,2)$ supersymmetric sigma model. Of course, one could envisage other twisting prescriptions, though the one formulated by Kapustin and Li seems to be the most natural. But this would hardly solve the discrepancy: indeed, there is no field in the $(2,2)$ supersymmetric sigma model, which, under twisting, may turn into the χ_a field of the biKähler sigma model. As we have seen, χ_a decouples in the A model, but it does not so in general.

[4]The analysis of ref. [11] is actually broader in scope, as it covers the more general case of a bihermitean target space and also considers a generalized B model.

[5]In ref. [11], it is shown that this cohomology is the total cohomology of a double cochain complex.

BiKähler moduli space. The biKähler sigma model is a topological field theory of cohomological type. It is known that any field theory like that describes the intersection theory of some moduli space in terms of local quantum field theory. On general grounds, the moduli space can be defined as the space of solutions of a set of equations, obtainable using the fixed point theorem of ref. [23]. For the biKähler sigma model, we derived these equations in section 5, see eqs. (5.12). However, we still do not have any geometrical interpretation or analytic understanding of this moduli space in general. Moduli spaces are notoriously subtle geometrical-topological structures. In order to be able to define intersection theory, one needs to compactify them and there is no unique of doing that in general. Moreover, they are usually plagued by singularities, which render them hardly amenable by standard means of analysis. A detailed investigation of these matters would be required.

BiKähler topological sectors and quantum cohomology. As is well known, the observables of A topological sigma model form a ring that is isomorphic to a deformation of the de Rham cohomology ring, going under the name of quantum cohomology [27 − 29]. This turns out to be an important invariant in symplectic geometry. The deformation is made possible by the fact that the model possesses non trivial topological sectors, with which there are associated world sheet instantons. Given the close relationship of the biKähler topological sigma model and the A model, it is natural to expect the biKähler model to be also characterized by a rich structure of topological sectors and world sheet instantons. A generalization of quantum cohomology would emerge in this way.

Since biKähler geometry allows one to construct a large number of topological terms to be added to the sigma model action by hand, the range of possibilities in which the quantum deformation could be carried out in the biKähler model is far wider than that of the A model. So, as a preliminary step, it seems that a classification of the meaningful topological terms would be required.

A biKähler sigma model containing the B model. The authors of ref. [22] were able to obtain both the A and B topological sigma models by gauge fixing suitable actions satisfying the Batalin-Vilkovisky master equation associated with the appropriate odd symplectic form. While the A model has appeared in our analysis as a particular case of the biKähler model, the B model has been conspicuously absent. Presumably, the B model can be obtained by gauge fixing the generalized Kähler Hitchin model action S_{GK} in a way analogous to that followed in section 8. We have not been able to do that so far, due to our present limited understanding reality conditions of the fields and of the geometry of the appropriate field space lagrangian submanifold $\mathfrak{L}$. This is definitely an issue calling for further investigation.

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JHEP01(2006)041